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OPTIMAL CONTROL OF PERIODIC MOTIONS OF LINEAR SYSTEMS WITH IMPULSIVE ACTION *

V.V. ANISOVICH and B.I. KRIUKOV

A problem is examined of optimizing linear systems with impulsive action and with a quadratic performance index. Conditions are formulated for the existence of an optimal periodic control. The optimal controls are written in explicit form as functions of the phase coordinates and of solutions of auxiliary equations. Discrete periodic systems are closely related to such problems were studied in /1,2/, where necessary optimality conditions were obtained.

1. We consider a controllable periodic system of differential equations with impulsive action

$$x^{\cdot}(t) = Ax(t) + Bu(t) + f(t),\ t \neq t_k,\ k = 0, \pm 1, \pm 2, \ldots$$
$$\Delta x|_{t=t_k} = x(t_k + 0) - x(t_k - 0) = \nu_k,\ \nu_k = (\nu_k^1, \ldots, \nu_k^n),\ \nu_k = \text{const} \tag{1.1}$$

Here $x(t)$ is an n-dimensional vector-valued function, $u(t)$ is an m-dimensional T-periodic piecewise-continuous vector-valued function, A and B are constant matrices of dimensions $n \times n$ and $n \times m$, respectively, $f(t)$ is a continuous n-dimensional T-periodic vector-valued function, for some positive integer p and a sequence of instants t_k the quantities $\nu_k = \nu_{k+p}$, $t_{k+p} = t_k + T$ are numbered by a set of integers such that $t_k \to -\infty$ as $k \to -\infty$ and $t_k \to +\infty$ as $k \to +\infty$. We assume that the real parts of the eigenvalues of matrix A are nonzero and that a positive number θ can be found such that $t_{k+1} - t_k \geqslant \theta$ for all $k = 0, \pm 1, \pm 2, \ldots$. In this case system (1.1) has a unique T-periodic solution /3/. We are required to find a T-periodic piecewise-continuous control $\bar{u}(t)$ and the corresponding T-periodic solution $\bar{x}(t)$ of Eq.(1.1), such that the functional

$$J(u) = \int_0^T (u^*Ku + u^*L^*x + x^*Lu + x^*Mx)\, dt \tag{1.2}$$

(* is the sign of transposition) takes the smallest value. The matrices K, L, M are constant, where K is positive-definite and K and M are symmetric. The control $\bar{u}(t)$ found thus is called optimal.

Using the rule for the generalized differentiation of piecewise-differentiable functions /4/, we obtain

$$x^{\cdot}(t) = [x^{\cdot}(t)] + \sum_{k=-\infty}^{+\infty} \{x(t_k + 0) - x(t_k - 0)\}\, \delta(t - t_k) \tag{1.3}$$

Here $x^{\cdot}(t)$ is the generalized derivative, $[x^{\cdot}(t)]$ is the usual derivative defined for any $t \neq t_k$. Since $[x^{\cdot}(t)] = Ax(t) + Bu(t) + f(t)$ when $t \neq t_k$, the original system (1.1) can be written as

$$x^{\cdot}(t) = Ax(t) + Bu(t) + f(t) + \Sigma(t),\quad \Sigma(t) = \sum_{k=-\infty}^{+\infty} \nu_k \delta(t - t_k) \tag{1.4}$$

Henceforth we assume that the equation

$$NBK^{-1}B^*N + N(BK^{-1}L^* - A) + (LK^{-1}B^* - A^*)N + LK^{-1}L^* - M = 0 \tag{1.5}$$

has a real symmetric solution N. We consider the system of vector differential equations

$$x^{\cdot}(t) = A_1x(t) + BK^{-1}B^*r(t) + f(t) + \Sigma(t)$$
$$r^{\cdot}(t) = -A_1^*r(t) + Nf(t) + N\Sigma(t) \tag{1.6}$$
$$A_1 = A - BK^{-1}L^* - BK^{-1}B^*N$$

and we assume that the real parts of the eigenvalues of matrix A_1 are nonzero. In this case system (1.6) has a unique T-periodic solution /3/.

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Theorem 1. When all the above-mentioned conditions are fulfilled the optimal control $\bar{u}(t)$ of problem (1.1), (1.2) exists and is computed by the formula

$$\begin{aligned} &\bar{u}(t) = -K^{-1}[(L^* + B^*N)\bar{x}(t) - B^*\bar{r}(t)], \quad t \neq t_k \\ &\bar{u}(t_k + 0) - \bar{u}(t_k - 0) = -K^{-1}L^*\nu_k \end{aligned} \tag{1.7}$$

where $\bar{x}(t)$, $\bar{r}(t)$ is a T-periodic solution of system (1.6).

Proof. We reduce functional (1.2) to canonic form. To do this we use (1.4) and (1.5) to implement the transformation

$$\begin{aligned} x^*Mx = {} & x^*NBK^{-1}B^*Nx + x^*NBK^{-1}L^*x + x^*NBu + x^*Nf + \\ & x^*N\Sigma(t) + x^*LK^{-1}B^*Nx + u^*B^*Nx + f^*Nx + [\Sigma(t)]^*Nx + \\ & x^*LK^{-1}L^*x - d(x^*Nx)/dt \end{aligned} \tag{1.8}$$

Substituting into (1.8) the expression for $Nf + N\Sigma(t)$ found from the second equation of system (1.6) and the Ax from (1.1) and using the T-periodicity of functions r and x, we write functional (1.2) in the canonic form

$$\begin{aligned} &J(u) = \int_0^T G^*KG\,dt - \int_0^T (r^*BK^{-1}B^*r + f^*r + r^*f)\,dt - \int_0^T (\Sigma^*(t)\,r + r^*\Sigma(t))\,dt \\ &G = u + K^{-1}L^*x + K^{-1}B^*Nx - K^{-1}B^*r \end{aligned} \tag{1.9}$$

Because matrix K is positive-definite the functional (1.9) has a minimum when $G = 0$, i.e., the minimum is reached for the value $u = \bar{u}(t)$ computed by formula (1.7). Substituting (1.7) into (1.4), we obtain the first equation of system (1.6) for the determination of $\bar{x}(t)$. The presence of δ-functions in the equations of system (1.6) signifies that $x(t)$ and $r(t)$ change stepwise at instants $t = t_k$. Neglecting in the right-hand sides of systems (1.6) the summands not containing δ-functions, at the instants $t = t_k$ we can set $x^{\cdot}(t) = \nu_k\delta(t - t_k)$, $r^{\cdot}(t) = N\nu_k\,\delta(t - t_k)$ or

$$x(t_k + 0) - x(t_k - 0) = \nu_k, \quad r(t_k + 0) - r(t_k - 0) = N\nu_k \tag{1.10}$$

Using (1.10) we can compute the magnitude $\bar{u}(t_k + 0) - \bar{u}(t_k - 0)$ of the discontinuity in the optimal control (1.7). The theorem is proved.

2. Let there exist a controlled periodic process with inpulsive action

$$\begin{aligned} &x^{\cdot}(t) = Ax(t) + Bu(t) + Cv(t) + f(t), \quad t \neq t_k, \quad k = 0, \pm 1, \pm 2, \ldots \\ &\Delta x\,|_{t=t_k} = x(t_k + 0) - x(t_k - 0) = \nu_k \end{aligned} \tag{2.1}$$

whose behavior is determined by two players: the first acts on the process by a control $u(t)$, while the second, by a control $v(t)$. The players' own behaviors are estimated by the performance index

$$J(u, v) = \int_0^T (u^*Ku + u^*L^*x + x^*Lu + x^*Mx + v^*Rv)\,dt \tag{2.2}$$

and they strive to choose controls $\bar{u}(t)$ and $\bar{v}(t)$ so as to fulfill the condition

$$J(\bar{u}, \bar{v}) = \min_u \max_v J(u, v) \tag{2.3}$$

The $\bar{u}(t)$ and $\bar{v}(t)$ thus chosen are said to be the optimal control. Here A, B, K, L, M, f, u, x are the same as in Sect.1, $v(t)$ is a q-dimensional T-periodic piecewise-continuous vector-valued function, R is a constant symmetric negative-definite of dimensions $q \times q$-matrix.

We introduce a constant symmetric matrix N_1 which is a solution of the equation

$$N_1NCR^{-1}C^*NN_1 + N_1(NCR^{-1}C^* + A_1) + (CR^{-1}C^*N + A_1^*)N_1 + CR^{-1}C^* + BK^{-1}B^* = 0 \tag{2.4}$$

and an n-dimensional vector-valued function $r_1(t)$ which is a solution of the equation

$$\begin{aligned} &r_1^{\cdot}(t) = -A_2r_1(t) + N_1Nf(t) + N_1N\Sigma(t) \\ &A_2 = CR^{-1}C^*N + A_1^* + N_1NCR^{-1}C^*N \end{aligned} \tag{2.5}$$

where N is a solution of Eq.(1.5). In addition to the conditions stated in Sect.1 we assume that the real parts of the eigenvalues of matrix A_2 are nonzero. In this case Eq.(2.5) has a unique T-periodic solution /3/. Implementing transformations analogous to those in /5/ and arguing as in the proof of Theorem 1, we obtain the following statement.

Theorem 2. When all the above-mentioned conditions are fulfilled the optimal controls $\bar{u}(t)$ and $\bar{v}(t)$ of problem (2.1)—(2.3) exist and are computed by the formulas

$$\begin{aligned}
&\bar{u}(t) = -K^{-1}[(L^* + B^*N)\bar{x}(t) - B^*\bar{r}(t)],\ t \neq t_k \\
&\bar{u}(t_k + 0) - \bar{u}(t_k - 0) = -K^{-1}L^*\nu_k \\
&\bar{v}(t) = R^{-1}C^*[(E + NN_1)\bar{r}(t) - N\bar{r}_1(t)],\ t \neq t_k \\
&\bar{v}(t_k + 0) - \bar{v}(t_k - 0) = R^{-1}C^*N\nu_k
\end{aligned} \tag{2.6}$$

where $\bar{r}_1(t)$ is a T-periodic solution of Eq.(2.5) and $\bar{x}(t)$, $\bar{r}(t)$ is a T-periodic solution of the system

$$\begin{aligned}
&x^{\cdot}(t) = A_1x(t) + (BK^{-1}B^* + CR^{-1}C^* + CR^{-1}CNN_2)r(t) - \\
&\quad CR^{-1}C^*Nr_1(t) + f(t) + \Sigma(t) \\
&r^{\cdot}(t) = A_2{}^*r(t) - NCR^{-1}C^*Nr_1(t) + Nf(t) + N\Sigma(t)
\end{aligned} \tag{2.7}$$

We note that when $L = 0$ the optimal control $\bar{u}(t)$ in (1.7) and (2.6) is a continuous vector-valued function. It is evident as well that if the real parts of all the eigenvalues of matrix A_1 are strictly negative, then the solutions $\bar{x}(t)$ corresponding to the optimal controls in problems (1.1), (1.2) and (2.1)—(2.3) are asymptotically stable.

Note. It is well known /6/ that a linear-quadratic optimization problem leads to a matrix Riccati differential equation

$$N^{\cdot} = NBK^{-1}B^*N + N(BK^{-1}L^* - A) + (LK^{-1}B^* - A^*)N + LK^{-1}L^* - M \tag{2.8}$$

Assuming that Eq.(2.8) has a unique T-periodic solution $N(t)$ (under this condition $N(t)$ is a symmetric matrix /5/) and using Eq.(2.8) instead of (1.5), we can convince ourselves that expression (1.9) remains in force and, consequently, so does the assertion of Theorem 1.

Using the specific nature of the problem, here we have been able to use instead of Eq. (2.8) the simpler Eq.(1.5) whose solution is a constant matrix N. At the expense of this, the coefficient A_1 in system (1.6) also is a constant matrix, which enables us /3/ to formulate the existence conditions for the T-periodic solution $(x(t), r(t))$ of system (1.6). An analogous situation arises in the minimax problem (2.1)— (2.3).

Equations of form (1.5) and (2.4) have been used to study linear-quadratic optimization problems on an infinite time interval /6/. Such an analogy with the periodic case is natural since the solution found on a period can be continued onto the infinite time interval.

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